

GROUNDSTATES OF NONLINEAR CHOQUARD EQUATIONS: HARDY-LITTLEWOOD-SOBOLEV CRITICAL EXPONENT

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ABSTRACT. We consider nonlinear Choquard equation

$$-\Delta u + Vu = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $V \in L^\infty(\mathbb{R}^N)$ is an external potential and $I_\alpha(x)$ is the Riesz potential of order $\alpha \in (0, N)$. The power $\frac{\alpha}{N} + 1$ in the nonlocal part of the equation is critical with respect to the Hardy-Littlewood-Sobolev inequality. As a consequence, in the associated minimization problem a loss of compactness may occur. We prove that if $\liminf_{|x| \rightarrow \infty} (1 - V(x))|x|^2 > \frac{N^2(N-2)}{4(N+1)}$ then the equation has a nontrivial solution. We also discuss some necessary conditions for the existence of a solution. Our considerations are based on a concentration compactness argument and a nonlocal version of Brezis-Lieb lemma.

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1. INTRODUCTION AND RESULTS

We consider a nonlinear Choquard type equation

$$(\mathcal{P}) \quad -\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \in \mathbb{N}$, $\alpha \in (0, N)$, $p > 1$, $I_\alpha : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}$ is the Riesz potential of order $\alpha \in (0, N)$ defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{N/2} \Gamma(\frac{\alpha}{2}) |x|^{N-\alpha}},$$

and $V \in L^\infty(\mathbb{R}^N)$ is an external potential.

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For $N = 3$, $\alpha = 2$ and $p = 2$ equation (\mathcal{P}) is the *Choquard-Pekar equation* which goes back to the 1954's work by S. I. Pekar on quantum theory of a Polaron at rest [6, Section 2.1; 20] and to 1976's model of P. Choquard of an electron trapped in its own hole, in an approximation to Hartree-Fock theory of one-component plasma [8]. In the 1990's the same equation reemerged as a model of self-gravitating matter [7, 19] and is known in that context as the *Schrödinger-Newton equation*.

Mathematically, the existence and qualitative properties of solutions of Choquard equation (\mathcal{P}) have been studied for a few decades by variational methods, see [8; 11; 12, Chapter III; 14] for earlier and [2–5, 13, 16–18] for recent work on the problem and further references therein.

The following sharp characterisation of the existence and nonexistence of nontrivial solutions of (\mathcal{P}) in the case of constant potential V can be found in [16].

Theorem 1 (Ground states of (\mathcal{P}) with constant potential [16, theorems 1 and 2]). *Assume that $V \equiv 1$. Then (\mathcal{P}) has a nontrivial solution $u \in H^1(\mathbb{R}^N) \cap L^{\frac{2Np}{N-\alpha}}(\mathbb{R}^N)$ with $\nabla u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap L^{\frac{2Np}{N-\alpha}}_{\text{loc}}(\mathbb{R}^N)$ if and only if $p \in (\frac{\alpha}{N} + 1, \frac{N+\alpha}{(N-2)_+})$.*

If $p \in [\frac{\alpha}{N} + 1, \frac{N+\alpha}{(N-2)_+}]$ then $H^1(\mathbb{R}^N) \subset L^{\frac{2Np}{N-\alpha}}(\mathbb{R}^N)$ by the Sobolev inequality, and moreover, every H^1 -solution of (\mathcal{P}) belongs to $W^{2,p}_{\text{loc}}(\mathbb{R}^N)$ for any $p \geq 1$ by a regularity result in [17, proposition 3.1]. This implies that the Choquard equation (\mathcal{P}) with a positive constant potential has no H^1 -solutions at the end-points of the above existence interval.

In this note we are interested in the existence and nonexistence of solutions to (\mathcal{P}) with nonconstant potential V at the *lower critical exponent* $p = \frac{\alpha}{N} + 1$, that is, we consider the problem

$$(\mathcal{P}_*) \quad -\Delta u + Vu = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u \quad \text{in } \mathbb{R}^N.$$

The exponent $\frac{\alpha}{N} + 1$ is critical with respect to the Hardy-Littlewood-Sobolev inequality, which we recall here in a form of minimization problem

$$c_\infty = \inf \left\{ \int_{\mathbb{R}^N} |u|^2 \mid u \in L^2(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} = 1 \right\} > 0.$$

Theorem 2 (Optimal Hardy-Littlewood-Sobolev inequality [9, theorem 3.1; 10, theorem 4.3]). *The infimum c_∞ is achieved if and only if*

$$(1.1) \quad u(x) = C \left(\frac{\lambda}{\lambda^2 + |x - a|^2} \right)^{N/2},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $\lambda \in (0, \infty)$ are parameters.

The form of minimizers in theorem 2 suggests that a loss of compactness in (\mathcal{P}_*) may occur by translations and dilations.

In order to characterise the existence of nontrivial solutions for the lower critical Choquard equation (\mathcal{P}_*) we define the critical level

$$c_* = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \mid u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1} = 1 \right\}.$$

It can be checked directly that if $u \in H^1(\mathbb{R}^N)$ achieves the infimum c_* , then a multiple of the minimizer u is a weak solution of Choquard equation (\mathcal{P}_*) .

Using a Brezis-Lieb type lemma for Riesz potentials [16, lemma 2.4] and a concentration compactness argument (lemma 10), we establish our main abstract result.

Theorem 3 (Existence of a minimizer). *Assume that $V \in L^\infty(\mathbb{R}^N)$ and*

$$(1.2) \quad \liminf_{|x| \rightarrow \infty} V(x) \geq 1.$$

If $c_ < c_\infty$ then the infimum c_* is achieved and every minimizing sequence for c_* up to a subsequence converges strongly in $H^1(\mathbb{R}^N)$.*

The inequality for the existence of minimizers is sharp, as shown by the following lemma for constant potentials.

Lemma 4. *If $V \equiv 1$, then $c_* = c_\infty$.*

Since problem (\mathcal{P}_*) with $V \equiv 1$ has no H^1 -solutions, this shows that the strict inequality $c_* < c_\infty$ is indeed essential for the existence of a minimizer for c_* .

In fact, the strict inequality $c_* < c_\infty$ is necessary at least for the strong convergence of *all* minimizing sequences.

Proposition 5. *Let $V \in L^\infty(\mathbb{R}^N)$. If*

$$\limsup_{|x| \rightarrow \infty} V(x) \leq 1,$$

then

$$c_* \leq c_\infty.$$

In addition, if

$$c_* = c_\infty,$$

then there exists a minimizing sequence for c_ which converges weakly to 0 in $H^1(\mathbb{R}^N)$.*

Using Hardy-Littlewood-Sobolev minimizers (1.1) as a family of test functions for c_* , we establish a sufficient condition for the strict inequality $c_* < c_\infty$.

Theorem 6. *Let $V \in L^\infty(\mathbb{R}^N)$. If*

$$(1.3) \quad \liminf_{|x| \rightarrow \infty} (1 - V(x))|x|^2 > \frac{N^2(N-2)_+}{4(N+1)},$$

then $c_ < c_\infty$ and hence the infimum c_* is achieved.*

In particular, if $N = 1, 2$ then condition (1.3) reduces to

$$\liminf_{|x| \rightarrow \infty} (1 - V(x))|x|^2 > 0,$$

that is, the potential $1 - V$ should not decay to zero at infinity faster than the inverse square of $|x|$.

Employing a version of Pohožaev identity for Choquard equation (\mathcal{P}_*) (see proposition 11 below), we show that a certain control on the potential V is indeed necessary for the strict inequality $c_* < c_\infty$.

Proposition 7. *Let $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If*

$$(1.4) \quad \sup \left\{ \int_{\mathbb{R}^N} \frac{1}{2} (\nabla V(x)|x| |\varphi(x)|^2 dx \mid \varphi \in C_c^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |\nabla \varphi|^2 \leq 1 \right\} < 1,$$

then Choquard equation (\mathcal{P}_) does not have a nonzero solution $u \in H^1(\mathbb{R}^N) \cap W_{\text{loc}}^{2,2}(\mathbb{R}^N)$.*

In particular, combining (1.4) with Hardy's inequality on \mathbb{R}^N , we obtain a simple nonexistence criterion.

Proposition 8. *Let $N \geq 3$ and $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If for every $x \in \mathbb{R}^N$,*

$$(1.5) \quad \sup_{x \in \mathbb{R}^N} |x|^2 (\nabla V(x)|x|) < \frac{(N-2)^2}{2},$$

then Choquard equation (\mathcal{P}_) does not have a nonzero solution $u \in H^1(\mathbb{R}^N) \cap W_{\text{loc}}^{2,2}(\mathbb{R}^N)$.*

For example, for $N \geq 3$ and $\mu > 0$, we consider a model equation

$$(1.6) \quad -\Delta u + \left(1 - \frac{\mu}{1+|x|^2}\right)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u \quad \text{in } \mathbb{R}^N.$$

Then proposition 8 implies that (1.6) has no nontrivial solutions for $\mu < \frac{(N-2)^2}{4}$, while for $\mu > \frac{N^2(N-2)}{4(N+1)}$ assumption (1.3) is satisfied and hence (\mathcal{P}_*) admits a groundstate. We note that

$$\frac{\frac{(N-2)^2}{4}}{\frac{N^2(N-2)}{4(N+1)}} = 1 - \frac{N-2}{N^2},$$

so that the two bounds are asymptotically sharp when $N \rightarrow \infty$. We leave as an open question whether (1.6) admits a ground state for $\mu \in [\frac{(N-2)^2}{4}, \frac{N^2(N-2)}{4(N+1)}]$.

We emphasise that unlike the asymptotic sufficient existence condition (1.3), nonexistence condition (1.5) is a global condition on the whole of \mathbb{R}^N . For example, a direct computation shows that for $a = 0$ and every $\lambda > 0$, a multiple of the Hardy-Littlewood-Sobolev minimizer (1.1) solves the equation

$$(1.7) \quad -\Delta u + \left(1 + \frac{N(2|x|^2 - N\lambda^2)}{(|x|^2 + \lambda^2)^2}\right)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u \quad \text{in } \mathbb{R}^N.$$

Here (1.5) fails on an annulus centered at the origin, while $V(x) > 1$ and $(\nabla V(x)|x|) < 0$ for all $|x|$ sufficiently large. Moreover,

$$\lim_{|x| \rightarrow \infty} (1 - V(x))|x|^2 = -2N < 0 \leq \frac{N^2(N-2)_+}{4(N+1)}.$$

Note that the constructed solution u_λ satisfies

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V|u_\lambda|^2 = 0.$$

In particular, we are unable to conclude that $c_* < c_\infty$. We do not know whether u_λ is a groundstate of (1.7). However, if u_λ was not a groundstate, then we would have $c_* < c_\infty$ and (1.7) would then have a groundstate by theorem 3.

2. EXISTENCE OF MINIMIZERS UNDER STRICT INEQUALITY: PROOF OF THEOREM 3

In order to prove theorem 3 we will use a special case of the classical Brezis-Lieb lemma [1] for Riesz potentials.

Lemma 9 (Brezis-Lieb lemma for the Riesz potential [16, lemma 2.4]). *Let $N \in \mathbb{N}$, $\alpha \in (0, N)$, and $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^N)$. If $u_n \rightarrow u$ almost everywhere on \mathbb{R}^N as $n \rightarrow \infty$, then*

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1} &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1}) |u_n|^{\frac{\alpha}{N}+1} \\ &\quad - \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{\frac{\alpha}{N}+1}) |u_n - u|^{\frac{\alpha}{N}+1}. \end{aligned}$$

Our second result is a concentration type lemma.

Lemma 10. *Assume that $V \in L^\infty(\mathbb{R}^N)$ and $\liminf_{|x| \rightarrow \infty} V(x) \geq 1$. If the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)$ and converges in $L^2_{\text{loc}}(\mathbb{R}^N)$ to u as $n \rightarrow \infty$, then*

$$\int_{\mathbb{R}^N} V |u|^2 \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} V |u_n|^2 - \int_{\mathbb{R}^N} |u_n - u|^2.$$

Proof. Since the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)$ and converges in measure to u , we deduce by the Brezis-Lieb lemma [1] (see also [10, theorem 1.9]) that

$$\int_{\mathbb{R}^N} V |u|^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V |u_n|^2 - \int_{\mathbb{R}^N} V |u_n - u|^2.$$

Now, we observe that for every $R > 0$ and every $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} (1 - V) |u_n - u|^2 \leq \int_{B_R} (1 - V) |u_n - u|^2 + (1 - \inf_{\mathbb{R}^N \setminus B_R} V)_+ \int_{\mathbb{R}^N} |u_n - u|^2.$$

By the local $L^2_{\text{loc}}(\mathbb{R}^N)$ convergence, we note that

$$\lim_{n \rightarrow \infty} \int_{B_R} (1 - V) |u_n - u|^2 = 0.$$

Since $\lim_{R \rightarrow \infty} (1 - \inf_{\mathbb{R}^N \setminus B_R} V)_+ = 0$ and $(u_n - u)_{n \in \mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)$, we conclude that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} (1 - V) |u_n - u|^2 \leq 0;$$

the conclusion follows. \square

Proof of theorem 3. Let $(u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence for c_* , that is

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1}) |u_n|^{\frac{\alpha}{N}+1} = 1$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V |u_n|^2 \rightarrow c_*.$$

In view of our assumption (1.2) we observe that the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. So, there exists $u \in H^1(\mathbb{R}^N)$ such that, up to a subsequence, the sequence $(u_n)_{n \in \mathbb{N}}$ converges to u weakly in $H^1(\mathbb{R}^N)$ and, by the classical Rellich-Kondrachov

compactness theorem, strongly in $L^2_{\text{loc}}(\mathbb{R}^N)$. By the lower semi-continuity of the norm under weak convergence,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V|u_n|^2 = c_*.$$

and by Fatou's lemma

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{\alpha}{N}+1}) |u_n|^{\frac{\alpha}{N}+1} \leq 1.$$

In order to conclude, it suffices to prove that equality is achieved in the latter inequality.

We observe that by lemma 9,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{\frac{\alpha}{N}+1}) |u_n - u|^{\frac{\alpha}{N}+1} = 1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1}$$

while by lemma 10 and by the lower-semicontinuity of the norm under weak convergence,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} V|u_n|^2 - |u_n - u|^2 \\ (2.1) \quad &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V|u_n|^2 - |u_n - u|^2 \\ &= c_* - \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n - u|^2. \end{aligned}$$

By definition of c_∞ , we have

$$\int_{\mathbb{R}^N} |u_n - u|^2 \geq c_\infty \left(\int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{\frac{\alpha}{N}+1}) |u_n - u|^{\frac{\alpha}{N}+1} \right)^{\frac{N}{N+\alpha}}.$$

Therefore, we conclude that

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \leq c_* - c_\infty \left(1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1} \right)^{\frac{N}{N+\alpha}}.$$

In view of the definition of c_* this implies that

$$c_* \geq c_\infty \left(1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1} \right)^{\frac{N}{N+\alpha}} + c_* \left(\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1} \right)^{\frac{N}{N+\alpha}}.$$

Since by assumption $c_* < c_\infty$, we conclude that

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1} = 1,$$

and hence, by definition of c_* ,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 = c_*,$$

that is the infimum c_* is achieved at u . Moreover, from (2.1) we conclude that $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$. Since $V \in L^\infty(\mathbb{R}^N)$, this implies that $Vu_n \rightarrow Vu$ in $L^2(\mathbb{R}^N)$. Using (2.1) again, we conclude that

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2.$$

Since $(u_n)_{n \in \mathbb{N}}$ converges to u weakly in $H^1(\mathbb{R}^N)$, this implies that $(u_n)_{n \in \mathbb{N}}$ also converges to u strongly in $H^1(\mathbb{R}^N)$. \square

3. OPTIMALITY OF THE STRICT INEQUALITY

In this section we prove lemma 4 and proposition 5.

Proof of lemma 4. Let us denote by \tilde{c}_∞ the infimum on the right-hand side. By density of the space $H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ and by continuity in L^2 of the integral functionals involved in the definition of c_∞ , it is clear that $\tilde{c}_\infty \geq c_\infty$. We choose now $u \in H^1(\mathbb{R}^N)$ and define for $\lambda > 0$ the function $u_\lambda \in H^1(\mathbb{R}^N)$ for every $x \in \mathbb{R}^N$ by

$$u_\lambda(x) = \lambda^{N/2} u(\lambda x).$$

We compute for every $\lambda > 0$ that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^{\frac{\alpha}{N}+1}) |u_\lambda|^{\frac{\alpha}{N}+1} = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + |u_\lambda|^2 = \frac{1}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2.$$

Hence,

$$\inf_{\lambda > 0} \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + |u_\lambda|^2 = \int_{\mathbb{R}^N} |u|^2,$$

and we conclude that $\tilde{c}_\infty \leq c_\infty$. □

Proof of proposition 5. For $\lambda > 0$, let

$$u_\lambda(x) = C \left(\frac{\lambda}{\lambda^2 + |x|^2} \right)^{\frac{N}{2}} = \lambda^{-\frac{N}{2}} u_1\left(\frac{x}{\lambda}\right)$$

be a family of minimizers for c_∞ given in (1.1). We observe that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^{\frac{\alpha}{N}+1}) |u_\lambda|^{\frac{\alpha}{N}+1} = 1,$$

whereas by a change of variables,

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V |u_\lambda|^2 = \frac{1}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + \int_{\mathbb{R}^N} V\left(\frac{y}{\lambda}\right) \frac{C^2}{1 + |y|^2} dy.$$

By Lebesgue's dominated convergence theorem

$$\limsup_{\lambda \rightarrow 0} \int_{\mathbb{R}^N} V\left(\frac{y}{\lambda}\right) \frac{C^2}{1 + |y|^2} dy \leq \int_{\mathbb{R}^N} \frac{C^2}{1 + |y|^2} dy = c_\infty,$$

so we conclude that $c_* \leq c_\infty$. If, in addition, $c_* = c_\infty$ then for any $\lambda_n \rightarrow 0$, $(u_{\lambda_n})_{n \in \mathbb{N}}$ is a minimizing sequence for c_* , and the conclusion follows. □

4. SUFFICIENT CONDITIONS FOR THE STRICT INEQUALITY: PROOF OF THEOREM 6

For $a \in \mathbb{R}^N$ and $\lambda > 0$, let

$$u_\lambda(x) = C \left(\frac{\lambda}{\lambda^2 + |x - a|^2} \right)^{N/2}$$

be a family of minimizers for c_∞ as in (1.1). Then

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V |u_\lambda|^2 = c_\infty + \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \int_{\mathbb{R}^N} (V - 1) |u_\lambda|^2.$$

Denote

$$\mathcal{I}_V(a, \lambda) := \lambda^2 \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \lambda^2 \int_{\mathbb{R}^N} (V - 1)|u_\lambda|^2 < 0.$$

To obtain a sufficient conditions for $c_* < c_\infty$ it is enough to show that for some $a \in \mathbb{R}^N$,

$$(4.1) \quad \inf_{\lambda > 0} \mathcal{I}_V(a, \lambda) < 0,$$

Proof of theorem 6. If $N \leq 2$, then by (1.3) there exists $\mu > 0$ such that

$$\liminf_{|x| \rightarrow \infty} (1 - V(x))|x|^2 \geq \mu.$$

Therefore

$$\lim_{\lambda \rightarrow \infty} \lambda^2 \int_{\mathbb{R}^N} (1 - V)|u_\lambda|^2 = \lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^N} \frac{\lambda^2(1 - V(\lambda x))}{(1 + |x|^2)^N} dx \geq \int_{\mathbb{R}^N} \frac{\mu}{|x|^2(1 + |x|^2)^N} dx = \infty.$$

Since for every $\lambda > 0$,

$$\lambda^2 \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 = \int_{\mathbb{R}^N} |\nabla u_1|^2 < \infty,$$

the condition (4.1) is satisfied.

If $N \geq 3$, we observe that for every $\lambda > 0$,

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 = \frac{N^2(N-2)}{4(N+1)} \int_{\mathbb{R}^N} \frac{|u_\lambda(x)|^2}{|x|^2} dx.$$

This follows from the fact that

$$\int_{\mathbb{R}^N} \frac{|x|^2}{(1 + |x|^2)^{N+2}} dx = \frac{N-2}{4(N+1)} \int_{\mathbb{R}^N} \frac{1}{|x|^2(1 + |x|^2)^N} dx,$$

which can be proved by two successive integrations by parts. Then, after a transformation $x = \lambda y + a$,

$$\mathcal{I}_V(a, \lambda) = \int_{\mathbb{R}^N} \left(\frac{\frac{N^2(N-2)}{4(N+1)}}{|y|^2} - \lambda^2(1 - V(a + \lambda y)) \right) \frac{C^2}{(1 + |y|^2)^N} dy,$$

and in view of (1.3), sufficient condition is (4.1) is satisfied for $a = 0$, so we conclude that $c_* < c_\infty$. \square

Note that if the function $\lambda \mapsto \lambda^2(1 - V(a + \lambda y))$ is nondecreasing for every $y \in \mathbb{R}^N$, then $\lambda \mapsto \mathcal{I}_V(a, \lambda)$ is nonincreasing. Therefore $\mathcal{I}_V(a, \lambda)$ admits negative values if and only if it has a negative limit as $\lambda \rightarrow \infty$. The latter is ensured in theorem 6 via asymptotic condition (1.3). This explains that if the function $\lambda \mapsto \lambda^2(1 - V(a + \lambda y))$ is nondecreasing, like for instance, in the special case

$$V(x) = 1 - \frac{\mu}{1 + |x|^2},$$

then integral sufficient condition (4.1) is in fact equivalent to the asymptotic sufficient condition (1.3).

5. POHOŽAEV IDENTITY AND NECESSARY CONDITIONS FOR THE EXISTENCE

We establish a Pohožaev type identity, which extends the identities (5.1) obtained previously for constant potentials V [4, lemma 2.1; 15; 16, proposition 3.1; 17, theorem 3].

Proposition 11. *Let $N \geq 3$ and $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u \in W^{1,2}(\mathbb{R}^N)$. If*

$$\sup_{x \in \mathbb{R}^N} |(\nabla V(x)|x)| < \infty,$$

and $u \in W_{\text{loc}}^{2,2}(\mathbb{R}^N)$ satisfies Choquard equation (\mathcal{P}_) then*

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)|x)|u(x)|^2 dx.$$

Proof. We fix a cut-off function $\varphi \in C_c^1(\mathbb{R}^N)$ such that $\varphi = 1$ on B_1 and we test for $\lambda \in (0, \infty)$ the equation against the function $v_\lambda \in W^{1,2}(\mathbb{R}^N)$ defined for every $x \in \mathbb{R}^N$ by

$$v_\lambda(x) = \varphi(\lambda x)(\nabla u(x)|x)$$

to obtain the identity

$$\int_{\mathbb{R}^N} (\nabla u|\nabla v_\lambda) + \int_{\mathbb{R}^N} Vuv_\lambda = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}uv_\lambda.$$

We compute for every $\lambda > 0$, by definition of v_λ , the chain rule and by the Gauss integral formula,

$$\begin{aligned} \int_{\mathbb{R}^N} Vuv_\lambda &= \int_{\mathbb{R}^N} V(x)u(x)\varphi(\lambda x)(x|\nabla u(x)) dx \\ &= \int_{\mathbb{R}^N} V(x)\varphi(\lambda x)(x|\nabla(\frac{|u|^2}{2}))(x) dx \\ &= - \int_{\mathbb{R}^N} ((NV(x) + (\nabla V(x)|x))\varphi(\lambda x) + V(x)\lambda(x|\nabla\varphi(\lambda x))) \frac{|u(x)|^2}{2} dx. \end{aligned}$$

Since $\sup_{x \in \mathbb{R}^N} (\nabla V(x)|x) < \infty$, by Lebesgue's dominated convergence theorem it holds

$$\lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^N} Vuv_\lambda = -\frac{N}{2} \int_{\mathbb{R}^N} V|u|^2 - \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)|x)|u|^2.$$

By Lebesgue's dominated convergence again, since $u \in W^{1,2}(\mathbb{R}^N)$, we have (see [16, proof of proposition 3.1] for the details)

$$\lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^N} (\nabla u|\nabla v_\lambda) = -\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2.$$

and

$$\lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}uv_\lambda = -\frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}.$$

We have thus proved the Pohožaev type identity

$$\begin{aligned} (5.1) \quad \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} V|u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)|x)|u(x)|^2 dx \\ = \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}+1}. \end{aligned}$$

If we test the equation against u , we obtain the identity

$$\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V|u|^2 = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{N}+1}) |u|^{\frac{\alpha}{N}+1};$$

the combination of those two identities yields the conclusion. \square

Proof of propositions 7 and 8. Proposition 7 is a direct consequence of proposition 11, while proposition 8 follows from proposition 11 and the classical optimal Hardy inequality on \mathbb{R}^N ,

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2$$

which is valid for all $u \in H^1(\mathbb{R}^N)$ (see for example [21, theorem 6.4.10 and exercise 6.8]). \square

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